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EQUATIONS OF THE DYNAMICS OF SETS OF REACHABILITY IN PROBLEMS OF OPTIMIZATION AND CONTROL UNDER CONDITIONS OF UNCERTAINTY*

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An integral funnel equation is used to interpret the problem of control under conditions of uncertainty in terms of the dynamics of sets. Localization of this equation is obtained. The right-continuity of the bound of the set of reachability is proved, on the basis of which the dynamics of the set of reachability is reduced to an analysis of the local integral funnel equation at points of the bound of the sets of reachability. The local integral funnel equation reduces to a differential relation at the points of continuous differentiability of this bound, from which partial differential equations of the dynamics of the sets of reachability in the space of the positions, in the conjugate space and in the parametric form of the notation are obtained. A classification of these equations is given in accordance with the forms of representation of the sets and surfaces in Euclidean space. Bellman's well-known equation serves as a special case of the equation in the space of positions. A derivation of the maximum principle with a normalized conjugate system is presented for the boundary solutions. Its normalization eliminates the increase in the norm of the vector of conjugate variables and increases the time for the numerical calculations. The optimal control problem reduces to one of obtaining boundary solutions. A considerable number of papers (/1-5/, etc.) cover control under conditions of uncertainty.

1. Control under conditions of uncertainty as a problem of the dynamics of sets.

We consider the control system (the dot denotes differentiation with respect to time)

$$x^{\cdot} = f_*(t, x, v, u), \quad x \in R^n, \quad t_0 \leqslant t < T, \quad u \in U \subset R^m \tag{1.1}$$

Here v is the control vector, and $u = u(t)$ is the vector of perturbing influences, whose exact values a a priori unknown, but the bounds of whose possible values, specified by the set U, are known. The law of control is usually specified when designing a system in the form of a program $v = v(t)$ or regulator $v = v(t, x)$. Then the family W of trajectories $x(t)$, $t_0 \leqslant t \leqslant T$, corresponding to the different laws of variation of the indeterminate quantities $u(t) \in U$, $t_0 \leqslant t \leqslant T$, correspond to each initial value x_0 of system (1.1). We can characterize the family W obtained by its section, which consists of the possible values of the vectors of the positions of the system at the instant t

$$D(t) = D(t, t_0, x_0) = \{x(t){:}\ x(\cdot) \in W\}$$

The set $D(t)$ reflects the indeterminate form of the vector of the positions of the system, caused by the uncertainty of the law of variation $u(t)$. Knowing the evolution of $D(t)$, we can estimate the dynamic uncertainty of the state of the system, which is an important component of the analysis and synthesis of the system under conditions of uncertainty. In particular, if we specify the bounds $D_*(t)$ of permissible deviations of the state of the system from the prescribed law of their variation, caused by the perturbations $u(t)$, which is written in the form of the inclusion $D(t) \subset D_*(t)$, the synthesis of the regulator $v(t, x)$ is subject to the additional condition that this inclusion holds. Putting $f(t, x, u) = f_*(t, x, v(t, x), u)$, we rewrite (1.1) in the form

$$x^{\cdot} = f(t, x, u), \quad u \in U, \quad t_0 \leqslant t \leqslant T \tag{1.2}$$

Then u will be the vector of controls, and $D(t, t_0, x_0)$ will be the set of reachability of system (1.2).

2. The integral funnel equation.

We shall put $G(t, x) = \operatorname{conv} f(t, x, U)$, a convex envelope of the set $f(t, x, U)$ of permissible velocities of system (1.2) and shall consider the differential inclusion

$$x^{\cdot} \in G(t, x), \quad x \in R^n \tag{2.1}$$

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We shall denote /6/ by $K(R^n)$ the metric space of the compacta from R^n with the Hausdorff metric

$$\alpha(X_1, X_2) = \max\{\beta(X_1, X_2),\ \beta(X_2, X_1)\},\quad X_1, X_2 \in K(R^n)$$
$$\beta(X_1, X_2) = \max_{x\in X_1} d(x, X_2);\quad d(x, X_2) = \min_{x'\in X_2} \| x - x' \|$$
$$\| x \| = \langle x, x\rangle^{1/2},\quad \langle x, x'\rangle = x_1x_1' + \ldots + x_nx_n'$$

and shall denote by $\mathrm{Kv}(R^n)$ its subspace, which consists of convex compacta. Unless otherwise stated, we will further assume that $G(t, x)$ is a continuous mapping $R^{n+1} \to \mathrm{Kv}(R^n)$, which satisfies Lipschitz's local condition with respect to x, i.e. for any $(t, x) \in R^{n+1}$ $\varepsilon > 0$ and $L > 0$ will be obtained, such that $\alpha(G(t', x'), G(t', x'')) \leqslant L\| x' - x''\|$ when $\| x - x' \| < \varepsilon, \| x - x'' \| < \varepsilon$, $| t - t' | < \varepsilon$. We shall call the absolutely continuous function $x(t)$, $t_0 \leqslant t \leqslant T$, which satisfies (2.1) almost everywhere in $[t_0, T]$, its solution or trajectory.

The set of right-hand ends $x(T)$ of all possible solutions of (2.1) $x(t)$, $t_0 \leqslant t \leqslant T$ with the initial condition $x(t_0) \in A_0$ is called a set of reachability $A(T, t_0, A_0)$ with the initial set $A_0 \subset R^n$. Below, unless otherwise stated, by the solution we mean the solution of (2.1). Sets of reachability are an important characteristic of a controllable system and in aggregate are essentially another equivalent description of it. It is well-known /7/ that taking the convex envelope $\mathrm{conv}\, f(t, x, U)$ of a set of permissible velocities of system (1.2) gives the closure $\mathrm{cl}\, D(t, t_0, x_0)$ of its sets of reachability, i.e. $\mathrm{cl}\, D(t, t_0, x_0) = A(t, t_0, x_0)$. This enables us to reduce the analysis of $D(t, t_0, x_0)$ to an examination of $A(t, t_0, x_0)$.

An equation of the dynamics of the sets of reachability is obtained in /8, 9/:

$$\lim_{\sigma\to 0+} \sigma^{-1}\alpha(R_\sigma(t) R(t+\sigma)) = 0;\quad R_\sigma(t) \overset{\mathrm{def}}{=} \bigcup_{x\in R(t)} [x + \sigma G(t, x)] \tag{2.2}$$

and is called an integral funnel equation in /10-12/. Suppose $X(t_0) \in K(R^n)$. The continuous mapping $R(t) = R(t, t_0, X(t_0))$: $[t_0, t_0 + \eta) \to K(R^n)$, $\eta > 0$, $R(t_0) = X(t_0)$, which satisfies (2.2) when $t \in [t_0, t_0 + \eta)$, is called a solution of Eq.(2.2), or an R-solution with the initial set $X(t_0)$, produced by $G(t, x)$ /8, 9/. The ordinary differential equation in R^n is a special case of (2.2), and the theorems of existence, uniqueness and extendability of the solutions are transposed to (2.2) /8-12/. In particular, the unique R-solution $R(t, t_0, X(t_0))$, defined in the maximum interval of the existence $t_0 \leqslant t < \omega = \omega(t_0, X(t_0)) \leqslant \infty$, exists for $X(t_0) \in K(R^n)$.

Following /8, 9/, we will obtain the connection between the R-solutions and the sets of reachability.

Theorem 2.1. The equation $R(t, t_0, X(t_0)) = A(t, t_0, X(t_0))$ holds for $X(t_0) \in K(R^n)$ when $t_0 \leqslant t < \omega$.

Unless otherwise stated, we shall **everywhere** assume that $X(t)$ is a continuous mapping

$$X(\cdot)\colon [t_0, T) \to K(R^n) \tag{2.3}$$

The aim of this paper is to obtain the conditions for $X(t)$ and the sets of reachability to be identical, i.e.

$$X(t) = A(t, t_0, X(t_0)) \text{ when } t_0 \leqslant t < T \tag{2.4}$$

If $T > \omega$, it follows from Eq.(2.4) that the mapping (2.3) is discontinuous when $t = \omega$. Therefore Eq.(2.4) and the continuity of $X(t)$ guarantee the inclusion $T \in [t_0, \omega]$. From the previous analysis we will obtain the basic condition for (2.4).

Theorem 2.2. For Eq.(2.4) to hold, it is necessary and sufficient that the mapping (2.3) of $X(t)$ satisfies the integral funnel equation when $t_0 \leqslant t < T$

$$\lim_{\sigma\to 0+} \sigma^{-1}\alpha(X_\sigma(t), X(t+\sigma)) = 0;\quad X_\sigma(t) \overset{\mathrm{def}}{=} \bigcup_{x\in X(t)} [x + \sigma G(t, x)]$$

3. Localization of the integral funnel equation.

The localization of the conditions contained in the previous theorem is the basic content of this paper. For the set $X \subset R^n$ we use $V_\varepsilon(X)$, $\varepsilon > 0$ to denote its closed ε-neighbourhood in R^n. Consider some mappings $X_1(\cdot)$, $X_2(\cdot)$: $[0, \sigma') \to K(R^n)$, $X \in K(R^n)$, where the set X can be single-point. If $\varepsilon > 0$ and $\sigma_0 > 0$ is obtained for any $\mu > 0$, such that

$$X_1(\sigma) \cap V_\varepsilon(x) \subset V_{\sigma\mu}(X_2(\sigma)) \text{ when } 0 \leqslant \sigma < \sigma_0,\ x \in X \tag{3.1}$$
$$X_2(\sigma) \cap V_\varepsilon^\circ(x) \subset V_{\sigma\mu}(X_1(\sigma)) \text{ when } 0 \leqslant \sigma < \sigma_0,\ x \in X \tag{3.2}$$

we shall write

$$X_1(\sigma) \overset{X}{\simeq} X_2(\sigma) \tag{3.3}$$

We obtain the following results from the definitions.

Theorem 3.1. The ratio $\overset{X}{\simeq}$ is an equivalence ratio.

Theorem 3.2. The following conditions are equivalent for the continuous mappings $X_1(\sigma)$, $X_2(\sigma)$: $[0, \sigma') \to K(R^n)$, $X_1(0) = X_2(0)$:

$$(i) \quad \overline{\lim_{\sigma\to 0+}}\, \sigma^{-1}\alpha(X_1(\sigma), X_2(\sigma)) = 0, \qquad (ii) \quad X_1(\sigma) \overset{X_1(0)}{\simeq} X_2(\sigma)$$

Theorem 3.3. The following conditions are equivalent:
(*i*) (3.3) holds, (*ii*) the following relation holds

$$X_1(\sigma) \overset{x}{\simeq} X_2(\sigma) \text{ when } x \in X \tag{3.4}$$

Proof. The implication $(i) \Rightarrow (ii)$ is obvious. We shall prove the implication $(ii) \Rightarrow (i)$. We shall show that $\delta > 0$ and $\sigma_0 > 0$ will be obtained using $\mu > 0$, for which the following inclusion holds:

$$X_1(\sigma) \cap V_\delta(x) \subset V_{\sigma\mu}(X_2(\sigma)) \text{ for } 0 \leqslant \sigma < \sigma_0,\ x \in X \tag{3.5}$$

According to (3.4), $\varepsilon = \varepsilon(x) > 0$ and $\sigma_0 = \sigma_0(x) > 0$ will be obtained using $x \in X$, such that

$$X_1(\sigma) \cap V_{\varepsilon(x)}(x) \subset V_{\sigma\mu}(X_2(\sigma)) \text{ for } 0 \leqslant \sigma < \sigma_0(x) \tag{3.6}$$

We shall choose a finite subcovering $X \subset \{\bigcup V_i$ using $1 \leqslant i \leqslant k\}$, $V_i = V_{\varepsilon(x_i)}(x_i)$, $x_i \in X$ from the covering of the compactum $X \subset \{\bigcup V_{\varepsilon(x)}(x)$ using $x \in X\}$. Then $V_\delta(X) \subset \{\bigcup V_i$ using $1 \leqslant i \leqslant k\}$ for some $\delta \in (0, \varepsilon)$. Hence by virtue of (3.6) when $0 \leqslant \sigma \leqslant \sigma_0 = \min\{\sigma_0(x_1), \ldots, \sigma_0(x_k)\}$ the validity of the following inclusions follows:

$$X_1(\sigma) \cap V_\delta(X) \subset \bigcup_{1\leqslant i\leqslant k} (X_1(\sigma) \cap V_i) \subset V_{\sigma\mu}(X_2(\sigma))$$

which implies (3.5). We shall obtain the satisfaction of condition (3.1) apart from the notation. The satisfaction of condition (3.2) is proved in a similar way.

We obtain the first localized characteristic of the set of reachability from Theorems 2.2, 3.2 and 3.3.

Corollary 3.1. For Eq.(2.4) to hold, it is necessary and sufficient that

$$X_\sigma(t) \overset{x}{\simeq} X(t+\sigma) \text{ for } x \in X(t),\ t_0 \leqslant t < T \tag{3.7}$$

This result enables us to consider (3.7) as a local integral funnel equation at the point x. The set of reachability has the following important property. Any of its inner points belongs to the interior of the set of reachability during some time. Then Eq.(3.7) holds in a trivial way at the inner points of the set of reachability, and it is sufficient to consider Eq.(3.7) only at the boundary points. This will enable us to use partial differential equations later to describe the dynamics of the set of reachability. To give a mathematical formulation to these considerations, it is necessary to analyse the topological properties of the set of reachability.

4. Boundary solutions.

Theorem 4.1. Suppose $x(t)$, $t_0 \leqslant t \leqslant t_*$ is the solution of (2.1) and the converging sequences $t_i \to t_*$, $y_i \to x(t_*)$, $i \to \infty$ are obtained. Then i_0 will be sought and the sequence of solutions $y_i(t)$, $t_0 \leqslant t \leqslant t_i$, $i \geqslant i_0$ of the inclusion (2.1), such that $y_i(t_i) = y_i$ and

$$\max_{t_0\leqslant t\leqslant \min\{t_i, t_*\}} \| y_i(t) - x(t) \| \to 0 \quad \text{as} \quad i \to \infty$$

Proof. We shall define the vector-function b: $R^{2n+1} \to R^n$ using the condition

$$b(t, y, q) = \{b \in G(t, y): \| b - q \| = d(q, G(t, y))\}$$

By virtue of the convexity and continuity of $G(t, x)$ the function $b(t, y, q)$ is continuous. We shall define $b(t, y) = b(t, y, x^{\cdot}(t))$ and shall consider the differential equation

$$y^{\cdot} = b(t, y) \tag{4.1}$$

where the equation is understood almost everywhere. The conditions of the theorem of the existence of Caratheodory's solution hold for (4.1). The solution of (4.1) serves as a solution of (2.1). By virtue of the assumptions about $G(t, x)$ and the definition of $b(t, y)$ the solutions of Eq.(4.1) $y_i(t)$, $t_i - \eta_i < t \leqslant t_i, \eta_i > 0$ with the boundary conditions $y_i(t_i) = y_i$ satisfy an inequality of the Gronwall type for the deviation $\| y_i(t) - x(t) \|$. We can therefore define them when $t_0 \leqslant t \leqslant t_i$, starting from some $i = i_0$, and $y_i(t)$ satisfy the conclusion of the theorem.

Hence we obtain a number of corollaries using the indirect method. For $M \subset [t_0, \infty)$ we shall put

$$\{M, A(M, t_0, A_0)\} = \bigcup_{t\in M} (t, A(t, t_0, A_0)) \subset R^{n+1}$$

Corollary. 4.1. Suppose $A_0 \subset R^n$, $M \subset [t_0, \infty)$ are open sets. Then $\{M, A(M, t_0, A_0)\}$ is an open set.

4.2. Suppose $A_0 \subset R^n$ is an open set. Then $A(t_*, t_0, A_0)$ is open for $t_* \geqslant t_0$.

4.3. Suppose $x_i \to x_T \in A(T, t_0, X(t_0))$, $t_i \to T$ as $i \to \infty$, $x_i \notin A(t_i, t_0, X(t_0))$. Then any solution (2.1) $x(t), t_0 \leqslant t \leqslant T$ with the boundary conditions $x(t_0) \in X(t_0)$, $x(T) = x_T$ satisfies the inclusion

$$x(t) \in \partial A(t, t_0, X(t_0)) \text{ when } t_0 \leqslant t < T \tag{4.2}$$

4.4. Suppose $x_T \in A(T, t_0, X(t_0))$, $x_T \notin A(t, t_0, X(t_0))$ when $t_0 \leqslant t < T$. Then any solution of the inclusion (2.1) $x(t), t_0 \leqslant t \leqslant T$ with the boundary conditions $x(t_0) \in X(t_0)$, $x(T) = x_T$ satisfies the inclusion (4.2).

4.5. Suppose $x(t)$, $t_0 \leqslant t \leqslant T$ is the solution of (2.1), which satisfies the boundary conditions $x(t_0) \in X(t_0)$, $x(T) \in \partial A(T, t_0, X(t_0))$. Then $x(t)$ satisfies the inclusion (4.2).

We shall call the solution of (2.1) $x(t)$, $t_0 \leqslant t \leqslant T$, which satisfies (4.2), a boundary solution from the initial set $X(t_0)$, whilst it is not generally required that (4.2) holds when $t = T$.

5. Right-continuity of the bound of the set of reachability.

Consider the mapping (2.3).

Theorem 5.1. The following relation holds for $t \in [t_0, T)$:

$$\beta(\partial X(t), \partial X(t+\sigma)) \to 0 \text{ as } \sigma \to 0+$$

Proof. The continuity of $X(t)$ implies the convergence $\alpha(X(t), X(t+\sigma)) \to 0$ as $\sigma \to 0+$. Then $\sigma_0(x) > 0$ will be obtained with respect to $a > 0$ and $x \in \partial X(t)$, such that $V_a(x) \cap \partial X(t+\sigma) \neq \varnothing$ when $\sigma \in [0, \sigma_0(x))$. We shall isolate the finite subcovering $\partial X(t) \subset \{\bigcup V_a(x_i)$ with respect to $1 \leqslant i \leqslant k\}$ from the covering of the compactum $\partial X(t)$ by means of the sets $V_a(x)$, $x \in \partial X(t)$, and shall assume $\sigma_0 = \{\min \sigma_0(x_i)$ with respect to $1 \leqslant i \leqslant k\}$. Then $\beta(\partial X(t), \partial X(t+\sigma)) \leqslant 2a$ for $\sigma \in [0, \sigma_0)$. Hence the statement of the theorem follows by virtue of the arbitrariness of $a > 0$.

The continuity of $X(t)$ simplifies the verification of the right-continuity of the bound $\partial X(t)$.

Corollary 5.1. The bound $\partial X(t)$ is right-continuous at the instant $t \in [t_0, T)$, i.e. $\alpha(\partial X(t), \partial X(t+\sigma)) \to 0$ as $\sigma \to 0+$ when, and only when, $\beta(\partial X(t+\sigma), \partial X(t)) \to 0$ as $\sigma \to 0+$.

Theorem 5.2. The bound of the set of reachability $\partial A(t, t_0, X(t_0))$ is right-continuous in the set $[t_0, \omega)$.

Proof. The set of reachability $A(t) = A(t, t_0, X(t_0))$ is continuous in $[t_0, \omega)$ according to Theorem 2.1. Then by virtue of Corollary 5.1 it is sufficient to prove that

$$\beta(\partial A(t'+\sigma), \partial A(t')) \to 0 \text{ as } \sigma \to 0+,\ t' \in [t_0, \omega) \tag{5.1}$$

According to Corollary 4.5 and the definition of $A(t)$, the solution $x_\sigma(t) \in \partial A(t), t' \leqslant t \leqslant t' + \sigma$ of the inclusion (2.1) will be obtained for any point $x_\sigma \in \partial A(t'+\sigma)$, such that $x_\sigma(t'+\sigma) = x_\sigma$. It follows from the continuity of $A(t)$ and $G(t, x)$ that $\|x_\sigma(t'+\sigma) - x_\sigma(t')\| \to 0$ as $\sigma \to 0+$ uniformly with respect to $x_\sigma \in \partial A(t+\sigma)$, which gives (5.1).

Theorem 5.3. The following two conditions are equivalent:

(*i*) the bound $\partial X(t)$ is right-continuous at the instant $t' \in [t_0, T)$,

(*ii*) $\delta > 0$ will be obtained with respect to $\varepsilon > 0$, such that

$$X(t') \setminus V_\varepsilon(\partial X(t')) \subset X(t'+\sigma) \text{ for } \sigma \in [0, \delta) \tag{5.2}$$

Proof. $(i) \Rightarrow (ii)$. Assuming the opposite, we obtain $\sigma_i \to 0+$, $x_i \to x_\infty \in \operatorname{Int} X(t')$, $x_i \notin X(t+\sigma_i)$, $i \to \infty$. $x_i' \in X(t'+\sigma_i)$, $x_i' \to x_\infty$ will be obtained with respect to the continuity of $X(t)$. Then $x_i'' \to x_\infty$ $x_i'' \in \partial X(t'+\sigma_i)$ will be obtained. By virtue of (i) $x_\infty \in \partial X(t')$, which contradicts the inclusion $x_\infty \in \operatorname{Int} X(t')$.

$(ii) \Rightarrow (i)$. We shall assume the opposite. Then $\sigma_k \to 0+$, $k \to \infty$ and $\varepsilon > 0$ will be obtained by virtue of Corollary 5.1, such that $\beta(\partial X(t'+\sigma_k), \partial X(t')) > 3\varepsilon$. We can choose $x_i \in \partial X(t'+\sigma_i)$, $x_i \to x_\infty \in X(t')$, such that $V_{3\varepsilon}(x_i) \cap \partial X(t') = \varnothing$. We shall obtain $V_{2\varepsilon}(x_\infty) \cap \partial X(t') = \varnothing$. Hence $x_\infty \in \operatorname{Int} X(t')$ and, according to (ii), $V_\varepsilon(x_\infty) \subset X(t'+\sigma_i)$ when $\sigma_i \in [0, \delta)$, which contradicts the convergence $\sigma_i \to 0+$, $x_i \to x_\infty$, $x_i \in \partial X(t'+\sigma_i)$, $i \to \infty$.

6. Properties of the mapping of the linear approximation $\sigma \to X_\sigma(t)$.

Theorem 6.1. Suppose $V \in K(R^n)$, $\operatorname{Int} V \neq \varnothing$. Then $\sigma_0 = \sigma_0(t) > 0$ will be obtained, such that the set $X_\sigma \overset{\text{def}}{=} \{\bigcup [x + \sigma G(t, x)]$ with respect to $x \in X\}$ is open for any open set $X \subset V$ and $\sigma \in [0, \sigma_0)$.

Proof. $L > 0$ is obtained, such that

$$\alpha(G(t, x'), G(t, x'')) \leqslant L \|x' - x''\| \text{ when } x', x'' \in V$$

We will assume $\sigma_0 = 0{,}5\ L^{-1}$. We shall formulate $x_* = x_0 + \sigma q_0$, $x_0 \in X$, $q_0 \in G(t, x_0)$. It is sufficient to show that for $\sigma \in [0, \sigma_0)$ and small $\|x - x_*\|$ the equation $x = x' + \sigma q'$ is solvable with respect to $q' \in G(t, x'), x' \in X$. Suppose x_{i-1}, q_{i-1} are already defined. We assume $x_i = x - \sigma q_{i-1}$ and subject the choice $q_i \in G(t, x_i)$ to the condition: $\|q_i - q_{i-1}\| \leqslant \|q - q_{i-1}\|$ when $q \in G(t, x_i)$. Then $\|x_i - x_{i-1}\| \leqslant \sigma \|q_{i-1} - q_{i-2}\| \leqslant \sigma L \|x_{i-1} - x_{i-2}\|$. Hence $x_i \to x'$, $q_i \to q' \in G(t, x')$ as $i \to \infty$, $x = x' + \sigma q'$ and $\|x' - x_0\| \leqslant (1 - \sigma L)^{-1} \|x - x_*\| \leqslant 2 \|x - x_*\|$. This inequality and the openness of X imply the inclusion $x' \in X$ for small $\|x - x_*\|$.

Corollary 6.1. Suppose $V \in K(R^n)$, $\text{Int}\, V \neq \varnothing$. Then $\sigma_0 = \sigma_0(t) > 0$ will be obtained, such that the inclusions $x \in X \subset V$, $q \in G(t, x)$, $x + \sigma q \in \partial X_\sigma(t)$, $\sigma \in [0, \sigma_0)$ imply the inclusion $x \in \partial X$.

In the same way as Theorem 5.2 was obtained from Corollaries 4.5 and 5.1, from Corollaries 5.1 and 6.1 we obtain the characteristic form of the bound ∂X_σ.

Theorem 6.2. Suppose $X \in K(R^n)$. Then the bound ∂X_σ is right-continuous with respect to σ when $\sigma = 0$.

7. The fundamental localization theorem. *Theorem* 7.1. For (2.4) to hold, it is necessary and sufficient that the following conditions hold:

(*i*) the bound $\partial X(t)$ is right-continuous in the set $t \in [t_0, T)$;

(*ii*) the following relation holds:

$$X_\sigma(t) \overset{x}{\simeq} X(t + \sigma) \text{ when } x \in \partial X(t),\ t_0 \leqslant t < T \tag{7.1}$$

Proof. The necessity of conditions (*i*) and (*ii*) follows from Corollary 3.1 and Theorem 5.2. Conversely, suppose conditions (*i*) and (*ii*) hold. By virtue of (*i*) and Theorems 5.3 and 6.2, Eq. (3.7) holds when $x \in \text{Int}\, X(t)$. Hence we obtain the statement of the theorem, together with condition (*ii*) and by virtue of Corollary 3.1.

The rest of this paper will interpret this theorem, using the continuous differentiability of the bound of the reachable set.

8. Relation at a point of a continuously differentiable bound. In accordance with the natural representations, we will say that, in the neighbourhood of the point $x' \in \partial X(t')$, $t' \in [t_0, T)$ the bound $\partial X(t)$ is an $(n-1)$-dimensional C^1-surface which is continuously time-differentiable, if the bound $\partial X(t)$ can be represented using the followng **diffeomorphism** in the neighbourhood of x' for t close to t';

$$\varphi(t, z) = x' + z_1 e_1 + \ldots + z_{n-1} e_{n-1} + \rho(t, z) e_n, \quad z \in R^{n-1}, \quad \rho(t', 0) = 0 \tag{8.1}$$

i.e. (8.1) serves as the diffeomorphism $\partial X(t)$ in the neighbourhood x' to some neighbourhood of the origin of coordinates $z \in R^{n-1}$. We assume the function $\rho(t, z)$ is continuously differentiable; e_i, $1 \leqslant i \leqslant n$ are unit mutually orthogonal vectors, whilst e_i, $1 \leqslant i \leqslant n-1$ serve as tangents to, and e_n is a normal to, $\partial X(t')$ in x'. Hence it follows that the vectors $\varphi_{z_i} = \partial\varphi(t', 0)/\partial z_i$ are tangents to $\partial X(t')$ in x', whence by virtue of (8.1) we obtain

$$\rho_{z_i} = 0 \text{ when } 1 \leqslant i \leqslant n-1,\ t = t',\ z = 0 \tag{8.2}$$

If in addition $V_\varepsilon(x') \cap \text{Int}\, X(t') \neq \varnothing$ when $\varepsilon > 0$, then in the neighbourhood x' on one side of $\partial X(t')$ the interior $\text{Int}\, X(t')$ is arranged, and on the other - the addition $R^n \setminus X(t')$, whilst by virtue of the continuity of $X(t)$ this property is also preserved for t close to t'. In this case we will say that $\partial X(t')$ separates $\text{Int}\, X(t')$ and $R^n \setminus X(t')$ in the neighbourhood of x'.

We shall write $X(\cdot) \in \Lambda(t', x')$, if the conditions described above hold. Unless stated otherwise, we will assume that the normal e_n is external.

We shall fix $q_0 \in G(t', x')$, which satisfies the inequality

$$\langle q, e_n\rangle \leqslant \langle q_0, e_n\rangle \text{ when } q \in G(t', x') \tag{8.3}$$

Lemma 8.1. Suppose $X(\cdot) \in \Lambda(t', x')$. Then

$$|X(t') + \sigma G(t', x') \overset{x'}{\doteq} X(t') + \sigma q_0$$

$$|X_\sigma(t') \overset{x'}{\doteq} X(t') + \sigma G(t', x')$$

Proof. We have $X(t') + \sigma q_0 \subset X(t') + \sigma G(t', x')$. We shall indirectly prove that $\varepsilon > 0$ and $\sigma_0 > 0$ will be obtained with respect to $\mu > 0$, satisfying the inclusion

$$V_\varepsilon(x') \cap [X(t') + \sigma G(t', x')] \subset V_{\sigma\mu}(X(t') + \sigma q_0) \text{ when } 0 \leqslant \sigma < \sigma_0$$

whence follows the first statement of the lemma. We will assume that the sequence $\sigma_i \to 0+$, $\varepsilon_i \to 0+$, $x_i = x_i' + \sigma_i q_i'$, $\|x_i - x'\| \leqslant \varepsilon_i$, $x_i' \in X(t')$, $q_i' \in G(t', x')$, $x_i' \to x'$, $q_i' \to q \in G(t', x')$, $x_i \notin V_{\sigma_i\mu}(X(t') + \sigma_i q_0)$ is obtained. Then for some i_0 we have

$$x_i' + \sigma_i(q - q_0) \notin V_{\sigma_i\mu/2}(X(t')) \text{ when } i \geqslant i_0$$

which contradicts the continuous differentiability of $\partial X(t')$. The second statement of the lemma follows from the continuity of $G(t, x)$.

From Theorem 3.1 and Lemma 8.1 we have

Theorem 8.1. Suppose $X(\cdot) \in \Lambda(t', x')$. Then $X_\sigma(t') \overset{x'}{\simeq} X(t') + \sigma q_0$. From the definitions and continuous differentiability of $\partial X(t)$ there follows

Theorem 8.2. Suppose $X(\cdot) \in \Lambda(t', x')$. Then

$$X(t' + \sigma) \overset{x'}{\simeq} X(t') + \sigma \rho_t(t', 0)\, e_n; \ \rho_t(t', 0) = \partial\rho(t', 0)/\partial t$$

Note that the relation

$$X(t') + \sigma q_0 \overset{x'}{\simeq} X(t') + \sigma\rho_t(t', 0)\, e_n \ \text{ when } \ X(\cdot) \in \Lambda(t', x')$$

holds when, and only when,

$$\rho_t(t', 0) = \langle q_0, e_n \rangle \tag{8.4}$$

which, bearing in mind (8.3), we shall rewrite in the form

$$\rho_t(t', 0) = \max \langle q, e_n \rangle \ \text{ with respect to } \ q \in G(t', x') \tag{8.5}$$

Then from Theorems 3.1, 8.1 and 8.2 we obtain a basic localized statement for the continuously differentiable bound $\partial X(t)$.

Theorem 8.3. **Suppose $X(\cdot) \in \Lambda(t', x')$. Then the following conditions are equivalent:**

(*i*) $X_\sigma(t') \overset{x'}{\simeq} X(t' + \sigma)$, (*ii*) condition (8.5) holds.

By virtue of Theorem 7.1 the fundamental **relation** (8.5) indicates that the rate of displacement $\rho_t(t', 0)$ of the bound $\partial A(t)$ of the set of reachability $A(t)$ at the point $x' \in \partial A(t')$ in the direction of the unit external normal e_n equals the maximum projections on to e_n of permissible velocities q at this point.

9. The equation in state space.

We shall consider the continuous function $B(t, x)$: $R^{n+1} \to R$ and shall formulate $X(t) = \{x \in R^n: B(t, x) \leqslant 0\}$. Unless stated otherwise, as before we will assume that as definite $X(t)$ is a continuous mapping $[t_0, T) \to K(R^n)$, $T \in [t_0, \omega)$.

Suppose the point (t', x') satisfies the equation $B(t', x') = 0$ and the function $B(t, x)$ is continuously differentiable in the neighbourhood of (t', x'), whilst $\operatorname{grad} B(t', x') \equiv \operatorname{col}(B_{x_1}, \ldots, B_{x_n}) \neq 0$. **For fixed t' we shall denote by $\partial X^B(t')$ the set x', such that (t', x') satisfies the conditions formulated above. It is clear that** $\partial X^B(t') \subset \partial X(t')$ and the inclusion $x' \in \partial X^B(t')$ implies the inclusion $X(\cdot) \in \Lambda(t', x')$. According to (8.1), we can write

$$B(t' + \sigma, x' + e_n \rho(t' + \sigma, 0)) = 0, \ B(t', x') = 0$$

Differentiating with respect to σ and substituting (8.4) here, we obtain

$$B_t(t', x') + \langle \operatorname{grad} B(t', x'), \ e_n \rangle \langle q_0, e_n \rangle = 0$$

By virtue of the parallelism of e_n and $\operatorname{grad} B(t', x')$ this gives

$$B_t(t', x') + \langle \operatorname{grad} B(t', x'), \ q_0 \rangle = 0$$

Bearing in mind the definition of q_0 (8.3) and omitting the primes, here we finally obtain that when $x \in \partial X^B(t)$ the equation

$$B_t(t, x) + \max_{q \in G(t, x)} \langle \operatorname{grad} B(t, x), q \rangle = 0 \tag{9.1}$$

is equivalent to Eq.(8.5). Then from Theorems 7.1, 8.3 and the inclusion $X(\cdot) \in \Lambda(t, x)$ when $x \in \partial X^B(t)$ we have

Theorem 9.1. For (2.4) to hold it is necessary and sufficient that the following conditions hold:

(*i*) the bound $\partial X(t)$ is right-continuous in the set $t \in [t_0, T)$,

(*ii*) $X_\sigma \overset{x}{\simeq} X(t + \sigma)$ when $x \in \partial X(t) \setminus \partial X^B(t)$, $t \in [t_0, T)$,

(*iii*) when $x \in \partial X^B(t)$, $t \in [t_0, T)$ condition (9.1) holds.

Remarks. 9.1. Since the inclusion $x \in \partial X^B(t)$ using the definition $\partial X^B(t)$ implies the equation $B(t, x) = 0$, Eq.(9.1) is not considered outside the set of points (t, x), satisfying the equation $B(t, x) = 0$.

9.2. In the notation of system (1.2) the equation in the space of the states **(Eq.(9.1))** will be rewritten in the form

$$B_t + \max_{u \in U} \langle \operatorname{grad} B, f \rangle = 0; \ \ B = B(t, x), \ f = f(t, x, u) \tag{9.2}$$

Let us consider the system $x^{\cdot} = f(x, u)$, $f(0, 0) = 0, 0 \in U$, $u \in U$ and denote the minimum time necessary to reach point x from the origin of coordinates by $\Theta(x)$, i.e. $\Theta(x)$ is a Bellman function in the problem of transferring the system $x^{\cdot} = -f(x, u)$ to the origin of coordinates in the minimum time. Assuming $B(t, x) = \Theta(x) - t$, we obtain a Bellman equation from (9.2) for the problem of the limit speed of response

$$\max_{u \in U} \langle f(x, u), \operatorname{grad} \Theta(x) \rangle = 1$$

i.e. Bellman's equation serves as a special case of Eq.(9.2).

10. The equation in conjugate space. This equation was first obtained for convex sets of reachability /13, 14/, and its heuristic derivation was given in /15/ for non-convex sets of reachability. Here we need a generalization of the idea of a support function to non-convex sets /15, 16/. Suppose $X(\cdot) \in \Lambda(t', x')$. For $x \in \partial X(t)$ we shall denote the external normal to $\partial X(t)$ in x by $s(t, x) \neq 0$. Consider some open neighbourhoods $\varkappa$, $V^\circ(x')$, $V^\circ(s')$ for $t', x', s' = s(t', x')$ respectively. For $t \in \varkappa$, $s \in V^\circ(s')$ we shall use $x(t, s) \subset V^\circ(x')$ to denote the set of points (possibly empty) $x \in \partial X(t) \cap V^\circ(x')$, in which s serves as an external normal to $\partial X(t)$ in x. We shall use the formula

$$r(t, s) \overset{\text{def}}{=} \langle x(t, s), s \rangle = \bigcup_{x \in x(t, s)} \langle x, s \rangle \text{ when } t \in \varkappa, s \in V^\circ(s') \tag{10.1}$$

to define the supporting mapping (possibly multivalued) in the neighbourhood $\varkappa \times V^\circ(x') \times V^\circ(s')$ of the point (t', x', s') with values in R. The asterisk indicates transposition.

Theorem 10.1. Suppose $X(\cdot) \in \Lambda(t', x')$; the function $\rho(t, z)$ in the representation (8.1) is doubly continuously differentiable, whilst the matrix of the two partial derivatives is non-degenerate with respect to z

$$\det \rho_{zz^*}(t', 0) \neq 0 \tag{10.2}$$

Then the open neighbourhoods $\varkappa$, $V^\circ(x')$ and $V^\circ(s')$ will be obtained for t', x' and s', such that Eq.(10.1) defines a single-valued continuously differentiable support mapping with values in R, in the neighbourhood $\varkappa \times V^\circ(x') \times V^\circ(s')$ whilst

$$x(t, s) = r_s^*(t, s) \in V^\circ(x') \text{ when } t \in \varkappa,\ s \in V^\circ(s') \tag{10.3}$$

$$r_t(t', s') = \rho_t(t', 0) \| s' \| \tag{10.4}$$

Proof. The vectors $\varphi_{z_i}(t, z)$ are tangential to $\partial X(t)$ at the point $\varphi(t, z)$. Therefore the condition of orthogonality of s to $\partial X(t)$ in $\varphi(t, z)$ will be written as

$$\langle \varphi_{z_i}(t, z), s \rangle = 0 \text{ when } 1 \leqslant i \leqslant n-1 \tag{10.5}$$

According to (8.1), this condition will be written in the form

$$\langle e_n, s \rangle \rho_{z_i}(t, z) + \langle e_i, s \rangle = 0, \quad 1 \leqslant i \leqslant n-1 \tag{10.6}$$

By virtue of (8.2) the vector $z = 0$ serves as a solution of (10.6) when $s = s'$, $t = t'$. A unique continuously differentiable function $z(t, s)$, serving as the solution of (10.6), will be obtained using the theorem on the implicit function by virtue of condition (10.2) for t, s, close to t', s'. Hence the support mapping will be written in the form $r(t, z(t, s), s) = \langle \varphi(t, z(t, s)), s \rangle$. Differentiating it and bearing in mind that, according to (10.5), $\varphi_z^* s = 0$, $(\varphi_z z_s, s) = (z_s, \varphi_z^* s) = 0$, we obtain Eq.(10.3) for $x(t, s) = \varphi(t, s) = \varphi(t, z(t, s))$. Since the vectors e_n and s' differ only by a positive multiplier, $\langle e_i, s' \rangle = 0$ when $1 \leqslant i \leqslant n-1$, and from (8.1) we obtain $r(t, s') = \langle x', s' \rangle + \rho(t, z(t, s')) \| s' \|$. Hence and from (8.2) we obtain (10.4).

Bearing in mind (10.4), we shall write Eq.(8.4) in the form $r_t = \langle q_0, e_n \rangle \| s' \| = \langle q_0, s' \rangle$. Bearing in mind (8.3) and (10.3) and omitting the primes, we shall rewrite it in the form

$$r_t(t, s) = \max \langle q, s \rangle \text{ with respect to } q \in G(t, r_s^*(t, s)) \tag{10.7}$$

We shall denote by $\partial X^r(t')$ the set of points $x' \in \partial X(t')$, such that $X(\cdot) \in \Lambda(t', x')$, whilst the function $\rho(t, z)$ is doubly continuously differentiable and satisfies (10.2). Consider a single-valued support mapping $r(t, s)$, which is defined by Eq.(10.1) in some neighbourhood $\varkappa \times V^\circ(x') \times V^\circ(s')$ of the point (t', x', s'), $s' \neq 0$ and which satisfies the conclusion of Theorem 10.1. We shall denote the family of all such single-valued supporting mappings by $\{r(t, s)\}$. Then from Theorems 7.1, 8.3 and 10.1 we have

Theorem 10.2. For Eq.(2.4) to hold, it is necessary and sufficient that the following conditions hold:

(*i*) the bound $\partial X(t)$ is right-continuous in the set $t \in [t_0, T)$,

(*ii*) $X_\sigma \overset{x}{\simeq} X(t + \sigma)$ when $x \in \partial X(t) \setminus \partial X^r(t)$, $t \in [t_0, T)$,

(*iii*) the supporting mappings $\{r(t, s)\}$, $t \in [t_0, T)$ satisfy Eq.(10.7).

Remark 10.1. For system (1.2) Eq.(10.7) will be rewritten in the form /13-16/: $r_t = \max \langle f(t, r_s^*, u), s \rangle$ with respect to $u \in U$. Since the variable s is conjugate to x with respect to the scalar product $\langle x, s\rangle$, Eq.(10.7) is called an equation in conjugate space.

11. Equations in parametric form. We assume that $X(\cdot)\in\Lambda(t', x')$. Here it is convenient to write the diffeomorphism which specifies $\partial X(t)$ in the neighbourhood of the point $x'\in\partial X(t')$ for t close to t' in the more common form: $x = p(t, v)$, $v\in R^{n-1}$, $p(t, v)\in C^1$, $x' = p(t', v')$, whilst $p(t, v)$ when $t = \text{const}$, near to t', is a diffeomorphism of some neighbourhood of the point $v'\in R^{n-1}$ to $\partial X(t)$ in the neighbourhood x'. The diffeomorphism (8.1) is a special case of $p(t, v)$. On the other hand, the diffeomorphism $z\to v(z)$ will be obtained, reducing $p(t, v)$ to the form (8.1): $p(t, v(z)) = x' + z_1e_1 + \ldots + z_{n-1}e_{n-1} + \rho(t, z)\,e_n$. Hence (8.4) will be rewritten in the form $\langle p_t(t', v'), e_n\rangle = \langle q_0, e_n\rangle$, which in turn can be written in the form of the inclusion

$$p_t(t', v')\in\Psi(t, p(t', v'), e_n) \tag{11.1}$$

$$\Psi(t, x, s)\overset{\text{def}}{=}\{q\in R^n\colon \langle q, s\rangle = \max\langle q', s\rangle \text{ with respect to } q'\in G(t, x)\}$$

Having the matrix of partial derivatives $p_v(t, v)$, we can use some method to express the external normal $s(t, p(t, v))$ to $\partial X(t)$ at the point $p(t, v)$ by $p_v(t, v)$ using some continuous function h: $s(t, p(t, v)) = h(p_v(t, v))\neq 0$. Then omitting the primes, we rewrite (11.1) in the form of a differential inclusion in partial derivatives

$$p_t\in\Psi(t, p, h(p_v));\ p = p(t, v) \tag{11.2}$$

We shall denote by $\partial X^\Lambda(t)\subset\partial X(t)$ the set $x\in\partial X(t)$, such that $X(\cdot)\in\Lambda(t, x)$. We shall take some family $\{p(t, v)\}$, $t_0\leqslant t< T$, consisting of the above locally definite diffeomorphisms $p(t, v)$, such that any diffeomorphism specifies $\partial X(t)$ in the domain of its definition and any point $x\in\partial X^\Lambda(t)$, $t\in[t_0, T)$ belongs to the domain of values of at least one of the diffeomorphisms $p(t, v)$. Then from Theorems 7.1 and 8.3 we have

Theorem 11.1. For Eq.(2.4) to hold it is necessary and sufficient that the following conditions hold:

(i) the bound $\partial X(t)$ is right-continuous in the set $t\in[t_0, T)$,

(ii) $X_\sigma\overset{x}{\simeq}X(t+\sigma)$ when $x\in\partial X(t)\setminus\partial X^\Lambda(t)$, $t\in[t_0, T)$,

(iii) the mappings $p(t, v)$ of the family $\{p(t, v)\}$ satisfy (11.2).

We shall put $\psi(t, x, s) = \Psi(t, x, s)\cap G(t, x)$ and consider the set $\partial X^\psi(t)\subset\partial X^\Lambda(t)$ of points in which $\psi(t, p(t, v), h(p_v(t, v)))$ consists of a unique vector. Then the system

$$p_t = \psi(t, p, h(p_v)),\ p = p(t, v) \tag{11.3}$$

will be a special case of (11.2). Therefore from Theorems 7.1 and 11.1 we will obtain

Corollary 11.1. For Eq.(2.4) to hold it is sufficient that the following conditions hold:

(i) the bound $\partial X(t)$ is right-continuous in the set $t\in[t_0, T)$,

(ii) $X_\sigma\overset{x}{\simeq}X(t+\sigma)$ when $x\in\partial X(t)\setminus\partial X^\psi(t)$, $t\in[t_0, T)$,

(iii) the mappings $p(t, v)$ of the family $\{p(t, v)\}$ satisfy the partial differential Eqs.(11.3) when $p(t, v)\in\partial X^\psi(t)$.

Note that Eqs.(11.3) for (1.2) can be rewritten in the form

$$p_t = f(t, p, u(t, p, p_v)) \tag{11.4}$$

where $u(t, p, p_v)$ is obtained from condition (8.5):

$$\langle f(t, p, u(t, p, p_v)),\ h(p_v)\rangle = \max_{u\in U_i}\langle f(t, p, u),\ h(p_v)\rangle$$

It is obvious from (11.4) that $p(t) = p(t, v)$ is a limit trajectory when $v = \text{const}$.

12. Description of the dynamics of the set of reachability using boundary solutions. Considering a section at the instant $t\in[t_0, T)$ of the family $\{x(t)\}$, $t_0\leqslant t< T$ of all boundary solutions with the initial set $X(t_0)$, we will obtain the bound of the set of reachability $\partial A(t, t_0, X(t_0))$, which is characterised by the set of reachability $A(t) = A(t, t_0, X(t_0))$ itself. Therefore it is possible to describe the dynamics of the set of reachability using boundary solutions.

We shall obtain the necessary conditions of boundedness of the solutions. For $A(\cdot)\in\Lambda(t, x)$ we shall denote by $s(t, x)$ the external normal to $\partial A(t)$ at the point $x\in\partial A(t)$, which continuously depends on t, x. Then Theorems 7.1 and 8.3, which are definitions of the boundary solution and differentiability, give the following result.

Theorem 12.1. Suppose $x(t)$, $t_0\leqslant t< T$ is a boundary solution from the initial set $X(t_0)$; $t'\in(t_0, T)$; $A(\cdot)\in\Lambda(t', x(t'))$. Then $\varepsilon>0$ will be obtained, such that for all $t\in(t'-\varepsilon, t'+\varepsilon)$, for which the derivative $x_*(t)$ exists, the following condition holds:

$$\langle x^\cdot(t), s(t, x(t))\rangle = \max\langle q, s(t, x(t))\rangle \text{ with respect to } q\in G(t, x(t)) \tag{12.1}$$

Below we assume that U is a compactum and the function $f(t, x, u)$ in (1.2) is continuous together with the matrix $f_x(t, x, u)$ of its partial derivatives. This ensures that the above assumptions with respect to (2.1) hold. We shall denote by $\lambda(t, s)$ the arbitrary scalar function which is measurable when $s = \mathrm{const}$, continuous when $t = \mathrm{const}$, and for which the summed function $\lambda_*(t) > |\lambda(t, s)|$, $t \in [a, b]$, $s \in X$ will be obtained for any $X \in K(R^n)$ and $[a, b] \subset [t_0, T)$. We shall denote the $n \times n$-dimensional unit matrix by I. The following theorem shows that the evolution of the external normal $s(t) = s(t, x(t))$ can be described using the conjugate system of Pontryagin's maximum principle.

Theorem 12.2. (The maximum principle). Suppose $x(t)$, $t_0 \leqslant t < T$ is a boundary solution of (2.1) from the initial set $X(t_0)$, which satisfies (1.2) with some measurable control $u^\circ(t) \in U$; $t' \in (t_0, T)$; $A(\cdot) \in \Lambda(t', x(t'))$; $s' \neq 0$ is the external normal to $\partial A(t)$ at the point $x(t') \in \partial A(t')$. Then $\varepsilon > 0$ will be obtained, such that the condition

$$\langle f(t, x(t), u^\circ(t)), s(t)\rangle = \max \langle f(t, x(t), u), s(t)\rangle \text{ with respect to } u \in U \qquad (12.2)$$

where $s(t') = s'$, and $s(t)$ almost everywhere in $(t' - \varepsilon, t' + \varepsilon)$ satisfies the equation

$$s^\cdot = [\lambda(t, s) I - f_x^*]s; \quad f_x^* = f_x^*(t, x(t), u^\circ(t)) \qquad (12.3)$$

holds for $t \in (t' - \varepsilon, t' + \varepsilon)$, for which the derivative $x^\cdot(t)$ exists.

Proof. According to Theorem 12.1, it is sufficient to prove that the external normal $s(t, x(t))$ to $\partial A(t)$ in $x(t)$ when $t \in (t' - \varepsilon, t' + \varepsilon)$ is specified as the solution of (12.3) with $s(t') = s'$. Consider the solutions (1.2) $x_*(t)$ when $t \in [t', t' + \varepsilon)$ with the control $u = u^\circ(t)$ and the initial points $x_*(t') \in \partial A(t')$, which fill $\partial A(t')$ in the neighbourhood of $x(t')$. The points $x^\cdot(t)$ of these solutions will describe the $(n-1)$-dimensional C^1-surface $\Gamma(t)$, which passes through $x(t)$ and belongs to the set of reachability $A(t)$. Then by virtue of the continuous differentiability of $\partial A(t)$ in $x(t)$ the tangential hyperplanes $\tan \Gamma(t)$ and $\tan \partial A(t)$ to $\Gamma(t)$ and $\partial A(t)$ in $x(t)$ coincide. Therefore, it is sufficient to obtain a description of the evolution of the vector $s(t)$, which is normal to $\tan \Gamma(t)$. The generatrices $e_i(t)$ are generally non-unit, and the hyperplanes $\tan \Gamma(t)$ can be described using a system in variations

$$e_j^\cdot = f_x e_i, \quad 1 \leqslant i \leqslant n-1, \quad f_x = f_x(t, x(t), u^\circ(t))$$

The condition of the orthogonality of $s(t)$ to $e_i(t)$ is written as $\langle e_i(t), s(t)\rangle = 0$. By virtue of the relations $\langle e_i(t'), s'\rangle = 0, 1 \leqslant i \leqslant n-1$ differentiation of this system gives the sufficient condition for its satisfaction: $\langle f_x e_i(t), s(t)\rangle + \langle e_i(t), s^\cdot(t)\rangle = 0$, i.e. $\langle e_i(t), s^\cdot(t) + f_x^* s(t)\rangle = 0$, $t' \leqslant t \leqslant t' + \varepsilon$. This condition holds if $s^\cdot(t) + f_x^* s(t) = \lambda s(t)$, where λ is an arbitrary number, in particular $\lambda = \lambda(t, s)$, which gives (12.3).

Assuming $\lambda(t, s) = 0$, we obtain a standard formulation of the maximum principle. The presence of $\lambda(t, s)$ does not introduce fundamental differences, since it only affects the length of the vector $s(t)$. Specifying the different initial conditions $s(t_0)$, which correspond to the external normals to $\partial X(t_0)$ and $x(t_0) \in \partial X(t_0)$, and integrating (1.2), (12.3) in $[t_0, T)$ with the control obtained using (12.2), we can construct a family of boundary solutions. The exponential increase in $\|s(t)\|$ as t increases, which complicates the numerical calculations, is inherent in automatic control systems. Therefore the choice of $\lambda(t, s)$ can be subject to the additional condition $\|s(t)\| = \mathrm{const}$, i.e. $\langle s^\cdot(t), s(t)\rangle = 0$, whence we will obtain $\lambda = \langle f_x^* s, s\rangle \langle s, s\rangle^{-1}$. Substitution into (12.3) gives the normalized conjugate system

$$s^\cdot = [\langle f_x^* s, s\rangle \|s\|^{-2} I - f_x^*] s \qquad (12.4)$$

13. The optimal equation. We will say that $x(t)$, $t_0 \leqslant t \leqslant T$, $x(t_0) \in X(t_0)$ is an optimal trajectory - with respect to speed of response - from the initial set $X(t_0)$ if the trajectory $x_*(t)$, $t_0 \leqslant t \leqslant t_* < T$, does not exist, such that $x_*(t_0) \in X(t_0)$, $x(t_*) = x(T)$. From Corollary 4.4 we obtain the characteristic form of these trajectories.

Theorem 13.1. Suppose $x(t)$, $t_0 \leqslant t \leqslant T$ is an optimal trajectory - with respect to speed of response - from the initial set $X(t_0)$. It is then a limit trajectory from the initial set $X(t_0)$.

For the optimal control problem in its Lagrangian form the finite point of the optimal trajectory belongs to the bound of the set of reachability in an extended space /14/. Then, according to Corollary 4.5, the optimal trajectory is a limit trajectory. The above enables us to reduce the finding of optimal trajectories to obtaining limit trajectories.

14. The classification of equations of the set of reachability. We shall list the most common methods of **representing** sets: the pointwise representation, under which the set is characterized by the set of points of which it consists, without reduction to a description using the other characteristics; representation using an inequality; representation using a support function or a support mapping; representation by means of the parametric specification of the bound of the set.

Descriptions of the dynamics of the set of reachability using an integral funnel equation and partial differential equations in the space of states which is also conjugate to the space in parametric form, correspond to these forms. Partial differential equations were obtained using the notation of the basic relation (8.5) in terms of the chosen description of the surface $\partial A(t)$. Using other forms of describing $\partial A(t)$ and writing (8.5) in terms of them, we can also obtain other forms of partial differential equations of the dynamics of a set of reachability.

15. Comments. Bellman's generalized equation was obtained previously using the extension principle*, and was obtained in /17, 18/ by considering the bound of the integral funnel /19/. The use of localization of the integral funnel equation enabled us to construct a mathematical theory of Bellman's generalized equation with an indication of the necessary and sufficient conditions for Eq.(2.4) to hold. Bellman's equation for differential games was analysed in /3/. The function $B(t, x)$ in (9.2) can be defined irrespective of any connection with the optimal value of the criterion of quality, which is the fundamental difference between Eq.(9.2) and the usual Bellman equation.

A description of the bound of the set of reachability is considered in /19/ using trajectories that satisfy Pontryagin's maximum principle. R-solutions are useful to average the differential inclusions /20/. Corollary 4.5 reflects the integral funnel property found by Hukuhara (see /21, p.30/). The method of local relations which describe the dynamics of stable sets in a differential game was proposed in /22/. The generalization of the integral funnel equation to arbitrary locally compact metric spaces was considered in /23/. The simplest examples of the application of the above partial differential equations to calculating sets of reachability were discussed in /24/.

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REFERENCES

1. KRASOVSKII N.N., Game problems on the meeting of motions. Moscow, Nauka, 1970.
2. KURZHANSKII A.B., Control and supervision under conditions of uncertainty. Moscow, Nauka, 1977.
3. SUBBOTIN A.I. and CHENTSOV A.G., Optimizing the guarnatee in control problems. Moscow, Nauka, 1981.
4. OVSEEVICH A.I. and CHERNOUS'KO F.L., Bilateral estimates of the domains of reachability of controllable systems. PMM, 46, 5, 1982.
5. ATHANS M., KU R. and GERSHWIN S.B., The uncertainity threshold principle. IEEE Trans. Automat. Control, AC-22, 3, 1977.
6. BORISEVICH YU.G., GEL'MAN B.D., MYSHKIS A.D. and OBUKHOVSKII V.V., Multiple-valued mappings. In: Results in science and technology. Mathematical analysis. Moscow, VINITI, 19, 1982.
7. GAMKRELIDZE R.V., Slipping optimal modes. Dokl. AN SSSR, 143, 6, 1962.
8. PANASYUK A.I. and PANASYUK V.I., Asymptotic optimization of non-linear control systems. Minsk, Izd-vo Belorus. un-ta, 1977.
9. PANASYUK A.I. and PANASYUK V.I., An equation produced using a differential inclusion. Mat. Zametki, 27, 3, 1980.
10. TOLSTONOGOV A.A., The integral funnel equation of a differential inclusion. Mat. Zametki, 32, 6, 1982.
11. TOLSTONOGOV A.A., The problems of solutions of the integral funnel equation of a differential inclusion. Differents. Uravneniya, 20, 2, 1984.
12. TOLSTONOGOV A.A., The integral funnel equation of a differential inclusion in Banach space and the properties of its solutions. Dokl. AN SSSR, 276, 5, 1984.
13. PANASYUK A.I., REachability set differential equation and optimal control problem. - Math. Optimierung. Theorie und Anwendungen. Wartburg/Eisenach, DDR, 1981.
14. PANASYUK A.I., Equations of the domains of reachability and their use in optimal control problems. Avtomatika i Telemekhanika, 5, 1982.
15. PANASYUK A.I., Equation of sets of reachability. Sib. Mat. Zh., 25, 4, 1984.
16. PANASYUK A.I., A differential equation of non-convex sets of reqchability. Mat. Zametki, 37, 5, 1985.
17. BUTKOVSKII A.G., A differential-geometry method of structurally solving problems of controllability and finite control. Avtomatika i Telemekhanika, 1, 1982.
18. LEITMANN G., Otpimality and reachability with feedback control. In: Dynamica Systems and Microphysics. N.Y. L: Acad. Press, 1982.
19. BLAGODATSKIKH V.I. and FILIPPOV A.F., Differential inclusions and optimal control. Tr. Mat. in. AN SSSR im. V.A. Steklov, 169, 1985.

---

* Gurman V.I. and Konstantinov G.N. Sets of reachability of controllable systems. The connection with Bellman's equation. Irkutsk, 1981, 14p. Dep. VINITI, 14.08.81; No.4038-81.

20. PLOTNIKOV V.A., Averaging differential inclusions. Ukr. Mat. Zh., 31, 5, 1979.
21. HARTMAN F., Ordinary differential equations. Moscow, Mir, 1970.
22. GUSEINOV G.KH., SUBBOTIN A.I. and USHAKOV V.N., Derivatives of multiple-valued mappings and their application in game problems of control. Probl. Control Inform. Theory, 14, 3, 1985.
23. PANASYUK A.I., Quasidifferential equations in metric space. Differents. Uravneniya, 21, 8, 1985.
24. PANASYUK A.I. and PANASYUK V.I., Asymptotic main-line optimization of controllable systems. Minsk: Nauka i Tekhnika, 1986.

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OPTIMAL ESTIMATES OF THE COORDINATES OF SYSTEMS WITH A TIME LAG WITH RESPECT TO A SET OF CONTINUOUS AND DISCRETE OBSERVATIONS*

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Expressions for optimal estimates of coordinates of systems with a time lag when there are continuous and discrete measurements are established and investigated. The effect of the amount of lag on the quality of estimation is demonstrated as an example. The related problems of when there are only continuous measurements were considered previously /1, 2/.

1. Formulation of the problem. Consider a dynamic system whose motion in the segment $[0, T]$ is described by a stochastic equation with initial conditions

$$x^{\cdot}(t) = A(t)\, x(t-h_1) + \sigma_1(t)\, \xi_1^{\cdot}(t),\ 0 \leqslant t \leqslant T \tag{1.1}$$

$$x(s) = 0,\ s < 0;\ x(0) = x_0 \tag{1.2}$$

For system (1.1) we carry out the following continuous $y(t)$ and discrete y_i observations at specified instants of time t_i

$$y(t) = g(t)\, x(t-h) + \sigma_2(t)\, \xi_2^{\cdot}(t),\ \ 0 \leqslant t \leqslant T \tag{1.3}$$

$$y_i = \beta_i x(t_i) + r_i \zeta_i,\ 0 \leqslant t_i \leqslant T,\ i=1, \ldots, N;\ t_1 < t_2 < \ldots < t_N \tag{1.4}$$

In Eqs.(1.1)-(1.4) the phase vector $x \in R_n$ (where R_n is an n-dimensional Euclidean space); the matrices A, σ_1, g, σ_2 with piecewise-continuous elements and the matrices β_i and r_i are specified; the time-lag constants $h_1, h \geqslant 0$; the Gaussian random quantities with zero expectation and unit covariation matrix are denoted by ζ_i, and ξ_1 and ξ_2 are standard Wiener processes; the Gaussian random quantity x_0 is such that $Mx_0 = 0$, $D_0 = Mx_0x_0'$. Here M is the sign of expectation, the prime is the sign of transposition, and D_0 is a specified positive-definite matrix. The random quantities $\xi_1, \xi_2, x_0, \zeta_i$ are mutually independent. Finally, without loss of generality, it is assumed that $y \in R_n$, $y_i \in R_n$.

Note that consideration of the time lag in the channel of measurements (1.3) is caused by the finiteness of time necessary to carry out the observations and to work out their results.

The need to consider the time lag in a measurement channel has been noted repeatedly in applied work (e.g. /3/). The choice of the initial conditions in the form (1.2) indicates that the motion of the system is only described by Eqs. (1.1) for $t \geqslant 0$, and nothing is known regarding the system when $t < 0$. In accordance with this, the continuous observations (1.3) that can be produced in the segment of time $[0, h]$ cannot carry any data about the system either, which was reflected in the above assumptions.

The problem consists of constructing an estimate $m(T)$ - which is optimal in the mean-square sense - of the vector $x(T)$, using the results of the observations (1.3), (1.4) in the segment $[0, T]$. It is known that $m(T)$ is the conditional expectation $x(T)$ under the conditions

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